

How Well a Chaotic Quantum System Can Retain Memory of Its Initial State?

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(Dated: August 22, 2008)

In classical mechanics the local exponential instability effaces the memory of initial conditions and leads to practical irreversibility. In striking contrast, quantum mechanics appears to exhibit strong memory of the initial state. We relate the latter fact to the low (at most linear) rate with which the system's Wigner function gets during evolution more and more complicated structure and establish existence of a critical strength of external influence below which such a memory still survives.

PACS numbers: 05.45.Mt, 03.65.Sq, 05.45.Pq

Strong numerical evidence has been obtained [1] that the quantum evolution of classically chaotic systems is quite stable, in sharp contrast to the extreme sensitivity of classical dynamics to initial conditions and weak external perturbations. Being the very essence of classical dynamical chaos, this sensitivity results in rapid loss of memory and practical irreversibility of classical motion. Qualitatively, this crucial difference is explained by a much simpler structure of quantum states as compared to the extraordinary complexity of random and unpredictable classical trajectories [2, 3]. In more rigorous terms, chaotic classical systems are characterized by positive *algorithmic complexity* described by Lyapunov exponent. Unfortunately, being formulated in such a way the concept of complexity cannot be transferred, *sic et simpliciter*, to quantum mechanics, where the very notion of trajectory is irrelevant and there is no quantum analogue to the Lyapunov exponent. Therefore, at first glance, there exists no quantitative measure of comparative complexity of classical and quantum states of motion [4].

However, individual classical trajectories are, in essence, of minor interest if the motion is chaotic. They all are alike in this case and rather behavior of manifolds of them carries really valuable information. Therefore the methods of the phase space and the Liouville form of the classical mechanics become the most adequate. It is very important that, opposite to the classical trajectories, the classical phase space distribution and the Liouville equation have direct quantum analogs. Hence, a comparison between classical and quantum dynamics can be made by studying the evolutions in time of the classical and quantum phase space distributions expressed in similar canonical variables and both ruled by linear equations.

The paramount property of the classical dynamical chaos is the exponentially fast structuring of the system's phase space on finer and finer scales. On the contrary, degree of structuring of the corresponding quantum "distribution" is restricted by the quantization of the phase space. This makes Wigner function relatively "simple" as

compared to its classical counterpart. The great advantage of the phase space approach is also that one operates with distributions which can be presented in both classical as well as quantum cases in identical action - angle (I, θ) variables.

In practice, to explore stability or reversibility of motion one computes *fidelity* [5, 6] which characterizes the weighted-mean distance between two distributions evolving, for example, under two slightly different Hamiltonians. In this paper, we exploit a somewhat different aspect of the Peres fidelity. Namely, we use it to establish a quantitative relation between the degree of loss of memory on the one hand and complexity of the corresponding phase space distribution on the other hand. In particular, we show how *the number $\mathcal{M}(t)$ of θ -Fourier harmonics* (see below Eqs. (7, 8)) can be used as a suitable measure of complexity of a distribution at a given time t . In the case of classical chaotic dynamics, this number grows exponentially in time with the rate related to the rate of local exponential instability [8]. At the same time, this number increases only power-like if the motion is regular. Thus the rate of growth of the number of harmonics is, similar to the Lyapunov exponent, a characteristic measure of classical complexity. In the framework of the phase space approach, the number of harmonics of the Wigner function appears, contrary to the Lyapunov exponent, to be a relevant measure of complexity also in quantum case. In what follows, we examine the time behavior of this quantity and its relation to fidelity and reversibility properties. A detailed derivation of the results reported in this paper can be found in Ref. [9].

Let $\hat{H} \equiv H(\hat{a}^\dagger, \hat{a}; t) = H^{(0)}(\hat{n} = \hat{a}^\dagger \hat{a}) + H^{(1)}(\hat{a}^\dagger, \hat{a}; t)$ be the Hamiltonian of a generic nonlinear system with a bounded below discrete energy spectrum $E_n^{(0)} \geq 0$, which is driven by a time-dependent force of such a kind that the classical motion exhibits a transition from integrable to chaotic behavior when the strength of the driving force is increased. Here \hat{a}^\dagger, \hat{a} are the bosonic, creation-

annihilation operators: $[\hat{a}, \hat{a}^\dagger] = 1$. We use the method of c-number α -phase space borrowed from the quantum optics (see for example [10, 11]). It is basically built upon the basis of the coherent states $|\alpha\rangle = \hat{D}\left(\frac{\alpha}{\sqrt{\hbar}}\right)|0\rangle$ obtained from the ground state $|0\rangle$ of the unperturbed Hamiltonian with the help of the unitary displacement operator $\hat{D}(\lambda) = \exp(\lambda \hat{a}^\dagger - \lambda^* \hat{a})$. Here α is a complex phase space variable independent of the effective Planck's constant \hbar .

The Wigner function W in the α -phase plane is defined by the following Fourier transformation

$$W(\alpha^*, \alpha; t) = \frac{1}{\pi^2 \hbar} \int d^2 \eta e^{(\eta^* \frac{\alpha}{\sqrt{\hbar}} - \eta \frac{\alpha^*}{\sqrt{\hbar}})} \text{Tr} \left[\hat{\rho}(t) \hat{D}(\eta) \right], \quad (1)$$

where $\hat{\rho}$ is the density operator and the integration runs over the complex η -plane. The Wigner function is normalized to unity, $\int d^2 \alpha W(\alpha^*, \alpha; t) = 1$ and is real though, in general, not positive definite. It satisfies the evolution equation

$$i \frac{\partial}{\partial t} W(\alpha^*, \alpha; t) = \hat{\mathcal{L}}_q W(\alpha^*, \alpha; t), \quad (2)$$

with the Hermitian “quantum Liouville operator” $\hat{\mathcal{L}}_q$. This equation reduces in the case $\hbar = 0$ to the classical Liouville equation with respect to the canonical pair $\alpha, i\alpha^*$ with the classical Hamiltonian function being given by the diagonal matrix elements $H_c(\alpha^*, \alpha; t) = \langle \alpha | \hat{H}^{(N)}(\hat{a}^\dagger, \hat{a}) | \alpha \rangle$ of the normal form $\hat{H}^{(N)}$ of the quantum Hamiltonian operator. In other words, this function is obtained from the quantum Hamiltonian by substituting $\hat{a} \rightarrow \alpha/\sqrt{\hbar}$, $\hat{a}^\dagger \rightarrow \alpha^*/\sqrt{\hbar}$.

We define the harmonic's amplitudes $W_m(I)$ as the Fourier components of the Wigner function with respect to the angle variable θ introduced by the canonical transformation $\alpha = \sqrt{I} e^{-i\theta}$. The normalization condition reduces then to $\int_0^\infty dI W_0(I; t) = 1$ whereas the amplitudes of other harmonics are expressed in terms of the matrix elements $\langle n+m | \hat{\rho} | n \rangle$ of the density operator along the m th collateral diagonal as

$$W_m(I; t) = \frac{2}{\hbar} e^{-\frac{2}{\hbar} I} \sum_{n=0}^\infty (-1)^n \sqrt{\frac{n!}{(n+m)!}} \times (4I/\hbar)^{\frac{m}{2}} L_n^m(4I/\hbar) \langle n+m | \hat{\rho}(t) | n \rangle, \quad m \geq 0, \quad (3)$$

with L_n^m Laguerre polynomials and $W_{-m} = W_m^*$. The inverted relation reads

$$\langle n+m | \hat{\rho}(t) | n \rangle = (-1)^n 2 \sqrt{\frac{n!}{(n+m)!}} \times \int_0^\infty dI e^{-2\frac{I}{\hbar}} (4I/\hbar)^{\frac{m}{2}} L_n^m(4I/\hbar) W_m(I; t). \quad (4)$$

Aiming to connect the reversibility of the motion with the complexity of the Wigner function, we follow the approach developed in Ref. [12]. We consider first the forward evolution $\hat{\rho}(t) = \hat{U}(t) \hat{\rho}(0) \hat{U}^\dagger(t)$ of a simple initial (generally mixed) state $\hat{\rho}(0)$ up to some time $t = T$.

An instantaneous Hermitian perturbation $\xi \hat{V} \delta(t - T)$ with the intensity ξ is then applied which transforms the state $\hat{\rho}(T)$ into $\hat{\tilde{\rho}}(T, \xi) = \hat{P}(\xi) \hat{\rho}(T) \hat{P}^\dagger(\xi)$. The resulting transformation $\hat{P}(\xi) = e^{-i\xi \hat{V}}$ is unitary. For example this transformation is equivalent to the global rotation $W(I, \theta; T) \rightarrow W(I, \theta + \xi; T)$ by the angle ξ in the phase plane if the operator $\hat{V} = \hat{n}$. In particular, we use below an infinitesimal perturbation of such a kind to reveal complexity of the Wigner function at the instant T .

The new state $\hat{\tilde{\rho}}(T, \xi)$ serves as the initial condition for the backward evolution $\hat{U}(-T) = \hat{U}^\dagger(T)$ during the same time T , after which the reversed state

$$\hat{\rho}(0|T, \xi) = \hat{U}^\dagger(T) \hat{\tilde{\rho}}(T, \xi) \hat{U}(T) = \hat{P}(\xi, T) \hat{\rho}(0) \hat{P}^\dagger(\xi, T), \quad (5)$$

is finally obtained. Here $\hat{P}(\xi, T) \equiv e^{-i\xi \hat{V}(T)}$, with $\hat{V}(T) \equiv \hat{U}^\dagger(T) \hat{V} \hat{U}(T)$ being the Heisenberg evolution of the perturbation during the time T . At last, we consider the distance between the reversed $\hat{\rho}(0|T, \xi)$ and the initial $\hat{\rho}(0)$ states, as measured by the Peres fidelity $F_{rev}(\xi; T) = \text{Tr}[\hat{\rho}(0|T, \xi) \hat{\rho}(0)] / \text{Tr}[\hat{\rho}^2(0)]$ [5] which can be also expressed with the help of Eq. (4) in terms of the Wigner function as [6, 9]

$$F_{rev}(\xi; T) = \frac{\int d^2 \alpha W(\alpha^*, \alpha; 0) \tilde{W}(\alpha^*, \alpha; 0|T, \xi)}{\int d^2 \alpha W^2(\alpha^*, \alpha; 0)} = \frac{\int d^2 \alpha W(\alpha^*, \alpha; T) \tilde{W}(\alpha^*, \alpha; T, \xi)}{\int d^2 \alpha W^2(\alpha^*, \alpha; T)} \equiv F(\xi; T). \quad (6)$$

The fidelity is bounded in the interval $[0, 1]$ and is the closer to unity the more similar are the initial and reversed states. At the second line, which is a consequence of the unitary time evolution, the fidelity $F(\xi; T)$ measures the *complexity* of the Wigner function at the moment $t = T$ (see below). Both the functions $F_{rev}(\xi; T)$ and $F(\xi; T)$ are numerically identical. The relation (6) plays the key role in our analyses. It allows us not only to relate the degree of reversibility to the complexity of the state at the reversal time T but also to *establish a strong restriction on the upgrowth* of the number $\mathcal{M}(t)$ of harmonics of the Wigner function. The crucial point is that while in classical mechanics the number of Fourier components has no direct physical meaning, in quantum mechanics the number of the components of the Wigner function at any given time is related to the *degree of excitation* of the system (see for example eq. (12) below) and therefore unrestricted exponential growth of this number is not allowed [8, 15, 16]. It is important that the representation (6) is valid not only in the quantum but also in the classical case when the Wigner function reduces to the classical phase-space distribution function $W_c(\alpha^*, \alpha; t)$.

First of all, let us establish a connection between fidelity and complexity of the Wigner function at an arbitrary moment t . To do this we make the instant rotation $e^{-i\xi \hat{n}}$ at this moment and utilize then the expression

given in the second line in Eq. (6). This yields [9]

$$F(\xi; t) = 1 - 2 \sum_{m=1}^{\infty} \sin^2(\xi m/2) \mathcal{W}_m(t) = \frac{1 - \frac{1}{2} \xi^2 \langle m^2 \rangle_t + O(\xi^4)}{1 - \frac{1}{2} \xi^2 \langle m^2 \rangle_t + O(\xi^4)}, \quad (7)$$

where $\langle m^2 \rangle_t = \sum_{m=0}^{+\infty} m^2 \mathcal{W}_m(t)$ and

$$\mathcal{W}_{m \geq 0}(t) = \frac{(2 - \delta_{m0}) \int_0^{\infty} dI |W_m(I; t)|^2}{\sum_{m=-\infty}^{+\infty} \int_0^{\infty} dI |W_m(I; t)|^2}. \quad (8)$$

We define the number of developed by the time t harmonics of the Wigner function of a classically chaotic quantum system as $\mathcal{M}(t) = \sqrt{\langle m^2 \rangle_t}$ what is in line with the refs. [8, 16]. The set of positive definite quantities \mathcal{W}_m is normalized to unity, $\sum_{m=0}^{+\infty} \mathcal{W}_m = 1$, and can be therefore given a probabilistic interpretation. Notice that our definition of the number $\mathcal{M}(t)$ is applicable to classical dynamics as well, provided that the harmonics of the classical distribution function W_c instead of those of the Wigner function W are used in Eq. (8). Hence, the number of harmonics at arbitrary time t is defined via Peres fidelity as

$$\mathcal{M}^2(t) \equiv \langle m^2 \rangle_t = - \left. \frac{d^2 F(\xi; t)}{d\xi^2} \right|_{\xi=0} \quad (9)$$

As an illustrative example we consider further the kicked quartic oscillator defined by the Hamiltonian [15, 17, 18, 19]

$$\hat{H}(\hat{a}^\dagger, \hat{a}) = \hbar \omega_0 \hat{n} + \hbar^2 \hat{n}^2 - \sqrt{\hbar} g(t) (\hat{a} + \hat{a}^\dagger), \quad (10)$$

where $g(t) = g_0 \sum_s \delta(t - s)$. In our units, the time and parameters \hbar, ω_0 as well as the strength of the driving force are dimensionless. The period of the driving force is set to one. The classical dynamics of such an oscillator becomes chaotic when the kicking strength g_0 exceeds a critical value $g_{0,c} \approx 1$. The angular phase correlations decay in this case exponentially and the mean action grows diffusively with the diffusion coefficient $D \approx g_0^2$.

In the chosen model, the only difference between the classical and corresponding quantum Liouville operators consists in the substitution $(\omega_0 + 2|\alpha|^2 \Rightarrow \omega_0 - \hbar - \frac{1}{2} \hbar^2 \frac{\partial^2}{\partial \alpha^* \partial \alpha} + 2|\alpha|^2) \times (\alpha^* \partial / \partial \alpha^* - \alpha \partial / \partial \alpha)$ in the unperturbed ($g_0 = 0$) part $\mathcal{L}^{(0)}$. Opposite to the continuous function $2|\alpha|^2$, the spectrum of the operator $\hat{K} = -\frac{1}{2} \hbar^2 \frac{\partial^2}{\partial \alpha^* \partial \alpha} + 2|\alpha|^2$, which coincides formally with the Hamiltonian of a two-dimensional linear oscillator, is discrete. This reflects the quantization of the phase space which stays *ultima analysi* behind the much slower growth of the number of Harmonics of the Wigner function than that of the corresponding classical phase space distribution.

A numerical illustration of this statement is given in Fig. 1. The initial state is chosen to be a pure ground state $\hat{\rho}(0) = |0\rangle\langle 0|$ which corresponds to the isotropic Wigner function $W(\alpha^*, \alpha; 0) = 2 e^{-2|\alpha|^2}$ with the size

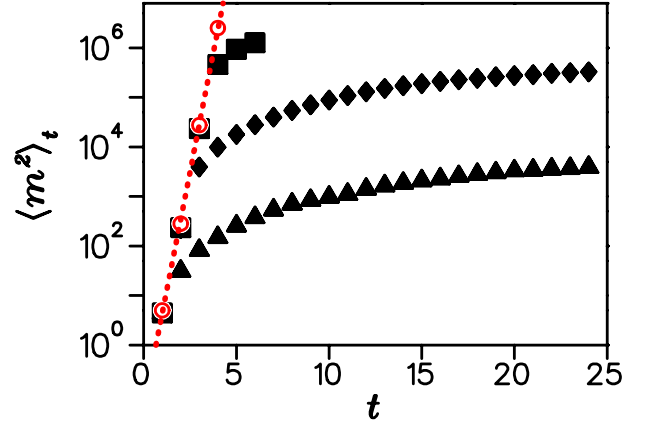


FIG. 1: (color online) Root-mean-square $\langle m^2 \rangle_t$ versus time t at $g_0 = 1.5$. Squares, diamonds and triangles correspond to $\hbar = 0.01, 0.1$ and 1 . Empty circles refer to classical dynamics and the dashed line fits these data.

$1/2$. This size is kept constant throughout all calculations whereas the quantum Liouville equation is solved for a number of decreasing values of the effective Planck's constant \hbar thus approaching the classical dynamics. Let us notice that for $\hbar < 1$ our initial condition corresponds to an incoherent mixture of eigenstates of the operator $\hat{H}^{(0)}$. It is clearly seen that the exponential increase of $\langle m^2 \rangle_t$ takes place only up to the Ehrenfest time scale $t_E \propto \ln \hbar$ [17], consistently with the findings reported in Refs. [8, 16]. Then, a much slower power-law increase follows.

To explain such a behavior we turn to the equivalent representation of fidelity given in the first line of Eq. (6). We restrict ourselves to the case of the pure ground initial state. More general consideration which includes also arbitrary incoherent initial mixtures is presented in [9]. Using this representation we find in this case that $F_{rev}(\xi, t) = |f(\xi, t)|^2$ where [9]

$$f(\xi, t) = \langle 0 | e^{-i\xi \hat{n}(t)} | 0 \rangle = \text{Tr} [e^{-i\xi \hat{n}} \rho(t)] = \frac{e^{i\xi/2}}{\cos(\xi/2)} \int_0^{\infty} e^{-2i \tan(\xi/2) \frac{I}{\hbar}} W_0(I; t). \quad (11)$$

In such a way we relate the behavior of fidelity to evolution of the action variable. Comparing now the ξ^2 terms in the expansions of the both possible representations (7) and (11) of fidelity we arrive at the following *significant exact relation* between the number of harmonics and the root-mean-square deviation of the action

$$\langle m^2 \rangle_t = 2 \chi_2(t), \quad \chi_2(t) \equiv \frac{1}{\hbar^2} (\langle I^2 \rangle_t - \langle I \rangle_t^2). \quad (12)$$

Thorough numerical study [9] convince us that after proper averaging over strong irregular fluctuations (*coarse graining*) the zero harmonic amplitude decays exponentially with the action I at any given time $t > t_E$,

$$\overline{W_0(I; t)} = \frac{1}{\langle I \rangle_t + \frac{\hbar}{2}} \exp \left(- \frac{I}{\langle I \rangle_t + \frac{\hbar}{2}} \right). \quad (13)$$

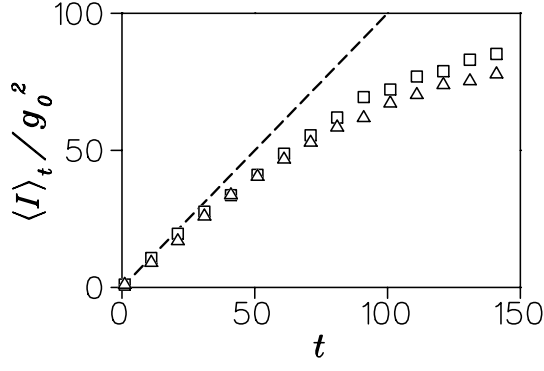


FIG. 2: Mean value $\langle I \rangle_t / g_0^2$ as a function of time t . Squares and triangles correspond to $(\hbar, g_0) = (1, 2)$ and $(2, 3)$. The straight line shows the classical diffusion law $\langle I \rangle_t = g_0^2 t$.

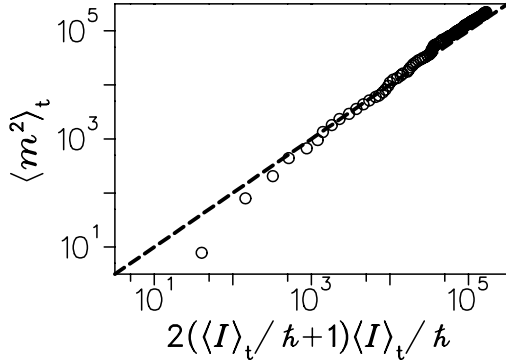


FIG. 3: $\langle m^2 \rangle_t$ vs $\chi_2(t) \approx \frac{\langle I \rangle_t}{\hbar} \left(\frac{\langle I \rangle_t}{\hbar} + 1 \right)$, for $t = 1, \dots, 100$. Data correspond to $\hbar = 1$ and $g_0 = 2$.

It follows from Eq. (13) that $\chi_2(t) \approx \frac{\langle I \rangle_t}{\hbar} \left(\frac{\langle I \rangle_t}{\hbar} + 1 \right)$ and, therefore, $\mathcal{M}(t) \approx \sqrt{2} \frac{\langle I \rangle_t}{\hbar}$. The time dependence of the mean action $\langle I \rangle_t$ (the deterministic quantum diffusion) is shown in Fig. 2 whereas the validity of the stated connection between the number of the harmonics and the excitation of the system is illustrated in Fig 3. Thus the number of harmonics grows after the Ehrenfest time not faster than linearly.

It can be readily shown now that for any finite $\xi \ll 1$ fidelity equals in the approximation (13)

$$F_{rev}(\xi; T) = F(\xi; t = T) = \frac{1}{1 + \frac{1}{2}\xi^2 \langle m^2 \rangle_T}. \quad (14)$$

More than that, this formula is valid for any time including the times $T < T_E$ [9]. The found result shows that a crossover takes place near the *critical value* $\xi_c(T) \equiv \sqrt{2}/\mathcal{M}(T)$, from good, $F(\xi; T) \approx 1$, to broken, $F(\xi; t) \approx (\xi/\xi_c(T))^2 \ll 1$, reversibility. Our numerical simulations (see Fig. 4) nicely confirm the formula (14).

The dependence of $\xi_c(T)$ on the reversal time T is strikingly different within and outside the Ehrenfest time scale. When in the former case it drops exponentially in accordance with the classical exponential proliferation of

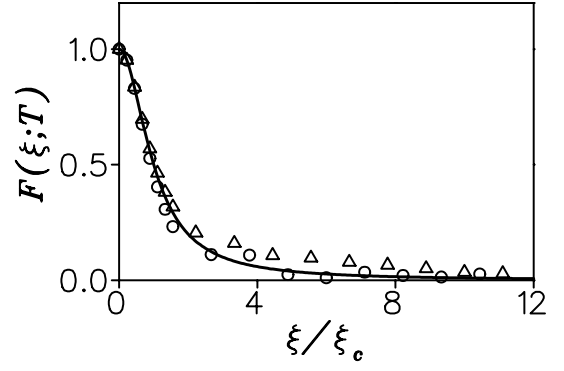


FIG. 4: Fidelity $F(\xi; T)$ versus the scaled variable $\xi/\xi_c(T)$. Data correspond to: (a) $\hbar = 1, g_0 = 2$; circles: $T = 10$; triangles: $T = 50$. The full curve shows the theoretical prediction Eq. (14). The deviations on the tail of the curve are due to fluctuations neglected in (14).

the number of harmonics, in the latter case it decreases at most linearly. This explains the numerically discovered [1] much weaker sensitivity of quantum dynamics to perturbations as compared to classical dynamics.

To summarize, we have established a quantitative relation between complexity of the Wigner function and degree of reversibility of motion of a classically chaotic quantum system. We have analytically proved that the number of harmonics $\mathcal{M}(t)$ of this function, which can serve as a natural measure of the complexity, increases after the Ehrenfest time *not faster than linearly* in striking contrast with classical dynamics, where the number of harmonics of phase space distribution grows exponentially. We have shown that if a quantum motion has been perturbed at some moment T by an external force with intensity ξ and then reversed, its initial state is recovered with the accuracy $\sim (\xi/\xi_c)^2$ as long as the strength is restricted to the interval $0 < \xi < \xi_c(T) = \sqrt{2}/\mathcal{M}(T)$. This interval decreases with the time T at most linearly beyond the semi-classical domain but shrinks exponentially due to the classical exponential instability when the this domain is approached.

The above outlined phase-space approach is quite general and can be readily extended to systems with arbitrary number of degrees of freedom, including qubit systems, whose Hamiltonian can be expressed in terms of a set of bosonic creation-annihilation operators. Therefore, this approach could shed some light of the connection between complexity and entanglement, a fundamental issue of great relevance for the prospects of quantum information science [20].

Acknowledgements. It is our pleasure to thank Giuliano Benenti for fruitful collaboration and illuminating discussions. We also are very indebted to Giulio Casati for drawing our attention to the problem of quantum reversibility. We acknowledge support by the Cariplo Foundation, INFN and by the RAS Joint scientific program

"Nonlinear dynamics and Solitons".

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- [1] D.L. Shepelyansky, *Physica D* **8**, 208 (1983); G. Casati, B.V. Chirikov, I. Guarneri, and D.L. Shepelyansky, *Phys. Rev. Lett.* **56**, 2437 (1986).
- [2] J. Ford, *Phys. Today*, April 1983, pag. 40.
- [3] V.M. Alekseev and M.V. Jacobson, *Phys. Rep.* **75**, 287 (1982).
- [4] T. Prosen, *J. Phys. A* **40**, 7881 (2007), and references therein.
- [5] A. Peres, *Phys. Rev. A* **30**, 1610 (1984).
- [6] T. Gorin, T. Prosen, T.H. Seligman, and M. Žnidarič, *Phys. Rep.* **435**, 33 (2006).
- [7] G. Benenti and G. Casati, *Phys. Rev. E* **65**, 066205 (2002);
- [8] A.K. Pattanayak and P. Brumer, *Phys. Rev. E* **56**, 5174 (1997); J. Gong and P. Brumer, *Phys. Rev. A* **68**, 062103 (2003).
- [9] V.V. Sokolov, O.V. Zhirov, G. Benenti, and G. Casati, preprint [arXiv:0807.2902v1 \[nlin.CD\]](#).
- [10] R.J. Glauber, *Phys. Rev.* **131** 2766 (1963).
- [11] G.S. Agarwal and E. Wolf, *Phys. Rev. D* **2**, 2161 (1970); *ibid.* 2187.
- [12] K.S. Ikeda, in *Quantum chaos: between order and disorder*, G. Casati and B.V. Chirikov (Eds.) (Cambridge University Press, 1995).
- [13] We note that, similarly to Ref. [14] and in contrast to other previous studies of the fidelity [6], the backward evolution proceeds with the same Hamiltonian as the forward evolution and the perturbation acts instantaneously only at the reversal time T .
- [14] C. Petitjean, D.V. Bevilacqua, E.J. Heller, and Ph. Jacquod, *Phys. Rev. Lett.* **98**, 164101 (2007).
- [15] B.V. Chirikov, F.M. Izrailev, and D.L. Shepelyansky, *Sov. Sci. Rev. C* **2**, 209 (1981).
- [16] Y. Gu, *Phys. Lett. A* **149**, 95 (1990).
- [17] G.P. Berman, G.M. Zaslavsky, *Physica A* **91**, 450 (1978); *ibid.* **97**, 367 (1979).
- [18] V.V. Sokolov, *Nonlinear Resonance of a Quantum oscillator* prep. Inst. Nucl. Phys. Siberia Div. Acad. Sci USSR (1978); *Teor. Mat. Fiz.* **61**, 128 (1984) [*Sov. J. Theor. Math.* **61**, 104 (1985)].
- [19] V.V. Sokolov, G. Benenti, and G. Casati, *Phys. Rev. E* **75**, 026213 (2007).
- [20] G. Benenti, preprint [arXiv:0807.4364v1 \[quant-ph\]](#) 28 Jul 2008.